Dr. Stefano Filipazzi
Dr. Alapan Mukhopadhyay
Léo Navarro Chafloque

EPFL, fall semester 2024
AG II - Schemes and sheaves

Exercises – week 4

Exercise 1. *Tangent vectors.* Let R be a ring, $R \to S$ an R-algebra and N an S-module. An *R-derivation*

$$d\colon S \to N$$

is an R-linear map such that for all $f, g \in S$

$$d(fg) = fd(g) + gd(f) \quad \text{(Leibniz rule)}$$

We denote this set by $\mathrm{Der}_R(S, N)$.

(1) Show that if $d\colon S \to N$ is an R-derivation, then $d(r) = 0$ for every $r \in R$.

Let $S \oplus_0 N$ denote the R-algebra with underlying R-module $S \oplus N$ and with the multiplication defined by

$$(s, n) \cdot (s', n') = (ss', sn' + s'n).$$

The multiplicative unit is $(1, 0)$.

(2) Show that the projection $S \oplus_0 N \to S$ is a ring map which has a square zero kernel ideal I, meaning that $I^2 = 0$.
(3) Show that the data of a derivation is the same as map of R-algebras $S \to S \oplus_0 N$ which is a section of the projection.

Let $X \to \mathrm{Spec}(k)$ a scheme over a field k.

(4) Show that $k[\epsilon] := k[t]/t^2$ is isomorphic to $k \oplus_0 k$.
(5) Let $x : \mathrm{Spec}(k) \to X$ be a k-rational point. We see $k(x)$ as an $\mathcal{O}_{X,x}$-algebra by the quotient map. Show that there are identifications

$$\mathrm{Der}_k(\mathcal{O}_{X,x}, k(x)) \cong \mathrm{Vect}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)) \cong \mathrm{Sch}_{k,x}(\mathrm{Spec}(k[\epsilon], X))$$

where $\mathrm{Sch}_{k,x}(\mathrm{Spec}(k[\epsilon], X))$ denotes k-schemes maps[1] that sends the point of $\mathrm{Spec}(k[\epsilon])$ to x.

Remark. Note that in differential geometry, what was exposed above is a way to define tangent spaces. See *Manifolds, sheaves and cohomology by Wedhorn, Remark 5.7* Therefore, in the context of the above exercise we define

$$(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$$

to be the be the *k-tangent space* of X at x.

Remark. We have just seen that maps of k-schemes from $\mathrm{Spec}(k[\epsilon])$ into a scheme are to be interpreted as a choice of a point and one tangent vector.

[1] meaning that the composition $\mathrm{Spec}(k[\epsilon]) \to X \to \mathrm{Spec}(k)$ is the one associated to $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ being the inclusion.

Therefore, we interpret $\mathrm{Spec}(k[\epsilon])$ as a point with a one dimensional tangent or *a point with an infinitesimal neighbourhood of order 1.*

Exercise 2. *Galois actions.* Let $\mathrm{Spec}(A) = X \to \mathrm{Spec}(k)$ an affine k-scheme where k is a field. Let $k \to l$ a Galois extension. We define on $\mathrm{Spec}(A \otimes_k l)$ an action of $\mathrm{Gal}(l\colon k)$ defined by $\phi_g = \mathrm{Spec}(\mathrm{id} \otimes g)$ where g is in $\mathrm{Gal}(l\colon k)$.

(1) For which $g \in \mathrm{Gal}(l\colon k)$ is the map ϕ_g a morphism of l-schemes ?
(2) Show that that the invariants[2] of $A \otimes_k l$ with this action is A.
(3) By the identification $\mathbb{C}[t] \cong \mathbb{R}[t] \otimes_{\mathbb{R}} \mathbb{C}$, show that the action of $\mathrm{Gal}(\mathbb{C}\colon \mathbb{R})$ on $\mathbb{C}[t]$ acts on the coefficients of a polynomial. For every $x \in \mathrm{Spec}(\mathbb{C}[t])$, what is $\phi_g(x)$ for the non-trivial $g \in \mathrm{Gal}(\mathbb{C}\colon \mathbb{R})$?
(4) Show that two points are identified by the natural map $\mathrm{Spec}(\mathbb{C}[t]) \to \mathrm{Spec}(\mathbb{R}[t])$ if and only if they are in the same orbit of the above action.
(5) What are the possible residue fields of points of $\mathrm{Spec}(\mathbb{R}[t])$? Show that the degree of the residue field at a closed point x as an extension of $\mathbb{R}$ corresponds to the cardinality of the fiber at x of the above map.

Remark. By duality, it means that $\mathrm{Spec}(A)$ is the quotient of $\mathrm{Spec}(A \otimes_k l)$ by the Galois action in the category of affine schemes. This can be useful to interpret what is a scheme over an arbitrary field. Namely, schemes over algebraically closed field are more *geometric* in nature and have more simpler properties – we can therefore interpret a scheme over any field as quotient by the absolute Galois group of k of a scheme over $\overline{k}$.

Exercise 3. *Noetherian topological spaces.* We say that a topological space is *Noetherian* if there is no infinite descending sequence of closed subsets in X. Show that if A is a Noetherian ring, then $\mathrm{Spec}(A)$ is Noetherian. Does the converse hold ?

Exercise 4. *Properties of maps.* Let $A \to B$ be a ring map. Denote by $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the associated map of schemes.

(1) Show that $A \to B$ is injective if and only $\mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_{\mathrm{Spec}(B)}$ is an injective map of sheaves.
(2) Show that if $A \to B$ is injective then the image of f is dense.
(3) Show that if $A \to B$ is surjective then f is a closed embedding on the underlying topological spaces.

Some notation. We introduce some notation needed for exercise 5 below. Let A and B be $\mathbb{N}$-graded rings. Let $d \geq 1$. We define the graded ring

$$A^{(d)} = \bigoplus_{n \geq 0} A_{nd}$$

[2]Elements of $A \otimes_k l$ fixed by the action of $\mathrm{Gal}(l\colon k)$.

and $v_d \colon A^{(d)} \to A$ for the canonical inclusion as a subring. This subring is called the d-Veronese subring.
We say that a ring map $\psi \colon A \to B$ is *homogeneous of degree* d if for $n \geq 0$ A_n maps to B_{dn}[3]. For example for the usual grading on $\mathbb{Z}[x]$ show that $x \mapsto x^d$ is homogeneous of degree d. Also, v_d is homogeneous of degree d.

Exercise 5. *Functoriality of* Proj. Let A and B be $\mathbb{N}$-graded rings. Let $\psi \colon A \to B$ be an homegeneous map of degree d for some $d \geq 1$.
To the contrary of Spec, the functoriality of Proj is not evident. The reason is that $\psi^{-1}(\mathfrak{p})$ for a prime $\mathfrak{p} \in \mathrm{Proj}(B)$ may contain the irrelevant ideal A_+.

(1) Show that
$$U(\psi) = \{\mathfrak{p} \in \mathrm{Proj}(B) \mid \psi(A_+) \not\subset \mathfrak{p}\}$$
is open. Namely show that it is the union of opens $D_+(\psi(f))$ for all homogeneous $f \in A_+$.
(2) Find an example where $U(\psi)$ is a non-empty open strict subspace of $\mathrm{Proj}(B)$.
(3) Show that ψ^{-1} defines a map of schemes $r_\psi \colon U(\psi) \to \mathrm{Proj}(A)$. Do this by defining a map $D_+(\psi((f)) \to D_+(f)$ for all homogeneous $f \in A_+$ and then glue.
(4) Show that if there exists a k_0 such that for all $k \geq k_0$ the map $A_k \to B_{dk}$ is surjective then $U(\psi) = \mathrm{Proj}(B)$. Show moreover that in this case r_ψ is a topological closed embedding with image $V_+(\ker(\psi))$.
(5) Show that if there exists a k_0 such that for all $k \geq k_0$ the map $A_k \to B_{dk}$ is an isomorphism then r_ψ is an isomorphism.
(6) Deduce that for any $d \geq 1$ and for $v_d \colon A^{(d)} \to A$ the map r_{v_d} is an isomorphism.

Exercise 6. *Dimension.* Let k be a field. Compute the irreducible components and their dimension of the spectrum of the following ring
$$k[x, y, z, t]/(tx, ty, tz).$$

Exercise to hand in. *Basic properties of* $\mathbb{P}^n_A$. (Due Sunday October 13th, 18:00)
Please write your solution in TEX.
Let A be a ring and $A[x_0, \ldots, x_n]$ the polynomial ring in $n+1$ variables over A. We define $\mathbb{P}^n_A \coloneqq \mathrm{Proj}(A[x_0, \ldots, x_n])$.

(1) Show that $D_+(x_i)$ for $i \in \{0, \ldots n\}$ provides an open cover of $\mathbb{P}^n_A$ and that each $D_+(x_i)$ is isomorphic to $\mathbb{A}^n_A$. Hint: use the result from class about homogeneous elements of degree 1.
Note: this point is telling us that projective n-space is obtained gluing $n+1$ copies of affine n-space, in the same way it is defined using varieties in classical algebraic geometry, or in complex or real geometry.

[3] Note that this means that the map factors through the d-Veronese subring.

(2) Show that $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$. Hint: use the covering of the previous point and the sheaf property.

Note: this is a first instance of a more general fact about projective varieties and can be thought of as an algebraic instance of the maximum modulus principle in complex analysis.

(3) Assume that $A = k$, where k is an algebraically closed field. Show that the closed points of $\mathbb{P}^n_k$ are identified with $(n+1)$-tuples $[a_0 : \ldots : a_n]$ satisfying the following properties:
- $a_i \in k$ for all i,
- not all a_i are 0, and
- two $(n+1)$-tuples $[a_0 : \ldots : a_n]$ and $[b_0 : \ldots : b_n]$ are identified if there exists $c \in k^*$ such that $b_i = c \cdot a_i$ for all i.

In other words, the points are identified with $(k^{n+1} \setminus 0)/k^\times$, *i.e.* linear subspaces of dimension 1 of k^{n+1}.

(4) Let $A = k$ be a field and B be a k-algebra. Show that every morphism of k-schemes $\mathbb{P}^n_k \to \mathrm{Spec}(B)$ is constant at the level of topological spaces with image a closed point which is k-rational. Hint: part (2) of this exercise.

(5) Let d be a positive integer and set $m := \binom{n+d}{d} - 1$. Use Exercise 5 and monomials of degree d to define an everywhere defined morphism[4], that we call a d-th Veronese embedding of $\mathbb{P}^n_A$

$$\psi_d \colon \mathbb{P}^n_A \to \mathbb{P}^m_A.$$

(6) Let k be an algebraically closed field. Describe the image of a second Veronese embedding $\psi_2 \colon \mathbb{P}^1_k \to \mathbb{P}^2_k$. Furthermore, using part (3), describe the closed points of the image as triples $[a_0 : a_1 : a_2]$. Find a homogeneous ideal I such that $\mathrm{im}(\psi_2) = V_+(I)$ – you can use Exercise 5 for that.

[4] Induce a map using Proj from an homogeneous map of degree d that sends monomials of degree 1 to all monomials of degree d.